Lienard-Wiechert Potential via Lorentz Transformation

The definition of $\overset{\Rightarrow}{A}$ makes derivation of the potentials of a moving point charge trivial. These are called the Lienard-Wiechert potentials and their derivation without special relativity is highly nontrivial, as seen in Griffiths §10.3.1

In the rest frame F of a point charge q, the covariant potential has components

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_o}\frac{q}{[x^2 + y^2 + z^2]^{1/2}} \qquad \vec{A}(\vec{r}, t) = 0 \tag{11.26}$$

Now, to have the charge move with velocity $\vec{v} = c\,\vec{\beta} = c\,\beta\,\widehat{x}$, we simply need to obtain $\overset{\Rightarrow}{A}$ in a lab frame $\widetilde{F}$ relative to which the charge's rest frame F moves at $\vec{v}$. That is, we use the Lorentz transformation, Equation 11.1, on $\overset{\Rightarrow}{A}$:

$$\frac{1}{c}\widetilde{V} = \gamma\left[\frac{1}{c}V + \beta A_x\right] = \frac{\gamma}{c}V \qquad \widetilde{A}_x = \gamma\left[A_x + \beta\frac{1}{c}V\right] = \frac{\gamma\beta}{c}V \tag{11.27}$$

and $\widetilde{A}_y = A_y = 0$ and $\widetilde{A}_z = A_z = 0$.

This is half the work. V is of course written in terms of the rest frame coordinates $r^\mu = (c\,t, x, y, z)$, so we need to rewrite it in terms of the lab frame coordinates $\widetilde{r}^\mu = (c\,\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{z})$. These coordinates are also related by Lorentz transformation, but we need the one going in the opposite direction (because we are now writing F coordinates in terms of $\widetilde{F}$ coordinates)

$$c\,t = \gamma\left[c\,\widetilde{t} - \beta\,\widetilde{x}\right] \qquad x = \gamma\left[-\beta\,c\,\widetilde{t} + \widetilde{x}\right] \tag{11.28}$$

and $y = \widetilde{y}$ and $z = \widetilde{z}$. (You can check that this is the correct direction for the transformation by considering the position of the F frame origin in $\widetilde{F}$ frame coordinates: $(x = 0, t)$ should obey $\widetilde{x} = \beta\,c\,\widetilde{t}$.) Combining the Lorentz transformation of the potential with the above transformation of the coordinates, we obtain:

$$\widetilde{V}(\widetilde{r}^\mu) = \frac{1}{4\pi\epsilon_o}\frac{\gamma\,q}{\left[\gamma^2\left(\beta\,c\,\widetilde{t} - \widetilde{x}\right)^2 + \widetilde{y}^2 + \widetilde{z}^2\right]^{1/2}} \tag{11.29}$$

$$= \frac{1}{4\pi\epsilon_o}\frac{q}{\left[\left\{\left(\widetilde{x} - v\,\widetilde{t}\right)^2 + \widetilde{y}^2 + \widetilde{z}^2\right\} - \beta^2\left(\widetilde{y}^2 + \widetilde{z}^2\right)\right]^{1/2}} \tag{11.30}$$

The coordinates in the above are the current position of the charge in $\widetilde{F}$.

Let's generalize the above expression for an arbitrary direction of motion by defining

$$R(\widetilde{r}^{\mu}) = \left[\left(\widetilde{x} - v\,\widetilde{t}\right)^2 + \widetilde{y}^2 + \widetilde{z}^2\right]^{1/2} \qquad \sin^2\theta = \frac{\widetilde{y}^2 + \widetilde{z}^2}{\left(\widetilde{x} - v\,\widetilde{t}\right)^2 + \widetilde{y}^2 + \widetilde{z}^2} \tag{11.31}$$

Then we have

$$\widetilde{V}(\widetilde{r}^{\mu}) = \frac{1}{4\,\pi\,\epsilon_o}\,\frac{q}{R(\widetilde{r}^{\mu})}\,\frac{1}{\left[1 - \beta^2\sin^2\theta\right]^{1/2}} \tag{11.32}$$

$$\widetilde{A}_i(\widetilde{r}^{\mu}) = \frac{1}{c^2}\,\frac{1}{4\,\pi\,\epsilon_o}\,\frac{q\,v_i}{R(\widetilde{r}^{\mu})}\,\frac{1}{\left[1 - \beta^2\sin^2\theta\right]^{1/2}} \tag{11.33}$$

where θ is now the angle between the direction of motion $\vec{v}$ and the vector $\vec{R}$ from the particle's current position to the position at which we want to know the potentials.

We will see later it will be useful to rewrite these expressions in terms of the retarded position of the particle — the position at the retarded time — but we will avoid that digression for now.